

Gorenstein locus of minuscule Schubert varieties

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Abstract

In this article, we describe explicitly the Gorenstein locus of all minuscule Schubert varieties. This proves a special case of a conjecture of A. Woo and A. Yong [WY06b] on the Gorenstein locus of Schubert varieties.

INTRODUCTION

The description of the singular locus and of the types of singularities appearing in Schubert varieties is a hard problem. A first step in this direction was the proof by V. Lakshmibai and B. Sandhya [LS90] of a pattern avoidance criterion for a Schubert variety in type A to be smooth. There exist some other results in this direction, for a detailed account see [BL00]. Another important result was a complete combinatorial description, still in type A , of the irreducible components of the singular locus of a Schubert variety (this has been realised, almost in the same time, by L. Manivel [Ma01a] and [Ma01b], S. Billey and G. Warrington [BW03], C. Kassel, A. Lascoux and C. Reutenauer [KLR03] and A. Cortez [Co03]). The singularity at a generic point of such a component is also given in [Ma01b] and [Co03]. However, as far as I know, this problem is still open for other types. Another partial result in this direction is the description of the irreducible components of the singular locus and of the generic singularity of minuscule and cominuscule Schubert varieties (see Definition 1.2) by M. Brion and P. Polo [BP99].

In the same vein as [LS90], A. Woo and A. Yong gave in [WY06a] and [WY06b] a generalised pattern avoidance criterion, in type A , to decide if a Schubert variety is Gorenstein. They do not describe the irreducible components of the Gorenstein locus but give the following conjecture (see Conjecture 6.7 in [WY06b]):

CONJECTURE 0.1. — *Let X be a Schubert variety, a point x in X is in the Gorenstein locus of X if and only if the generic point of any irreducible component of the singular locus of X containing x is in the Gorenstein locus of X .*

The interest of this conjecture relies on the fact that, at least in type A , the irreducible components of the singular locus and the singularity of a generic point of that component are well known. The conjecture would imply that one only needs to know the information on the irreducible components of the singular locus to get all the information on the Gorenstein locus.

In this paper we prove this conjecture for all minuscule Schubert varieties thanks to a combinatorial description of the Gorenstein locus of minuscule Schubert varieties. To do this we use the

combinatorial tool introduced in [Pe07] associating to any minuscule Schubert variety a reduced quiver generalising Young diagrams. First, we translate the results of M. Brion and P. Polo [BP99] in terms of the quiver. We define the holes, the virtual holes and the essential holes in the quiver (see Definitions 2.3 and 3.1) and prove the following:

THEOREM 0.2. — *(i) A minuscule schubert variety is smooth if and only if its associated quiver has no nonvirtual hole.*

(ii) The irreducible components of the singular locus of a minuscule Schubert variety are indexed by essential holes.

Furthermore we explicitly describe in terms of the quiver and the essential holes these irreducible components and the singularity at a generic point of a component (for more details see Theorem 3.2). In particular, with this description it is easy to say if the singularity at a generic point of an irreducible component of the singular locus is Gorenstein or not. The essential holes corresponding to irreducible components having a Gorenstein generic point are called Gorenstein holes (see also Definition 3.8). We give the following complete description of the Gorenstein locus:

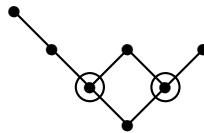
THEOREM 0.3. — *The generic point of a Schubert subvariety $X(w')$ of a minuscule Schubert variety $X(w)$ is in the Gorenstein locus if and only if the quiver of $X(w')$ contains all the non Gorenstein holes of the quiver of $X(w)$.*

COROLLARY 0.4. — *Conjecture 0.1 is true for all minuscule Schubert varieties.*

EXAMPLE 0.5. — *Let $\mathbb{G}(4, 7)$ be the Grassmannian variety of 4-dimensional subspaces in a 7-dimensional vector space. Consider the Schubert variety*

$$X(w) = \{V_4 \in \mathbb{G}(4, 7) \mid \dim(V_4 \cap W_3) \geq 2 \text{ and } \dim(V_4 \cap W_5) \geq 3\}$$

where W_3 and W_5 are fixed subspaces of dimension 3 and 5 respectively. The minimal length representative w is the permutation (2357146). Its quiver is the following one (all the arrows are going down):



We have circled the two holes on this quiver. The left hole is not a Gorenstein hole (this can be easily seen because the two peaks above this hole do not have the same height, see Definition 2.3) but the right one is Gorenstein (the two peaks have the same height). Let $X(w')$ be an irreducible component of the singular locus of $X(w)$. The possible quivers of such a variety $X(w')$ are the following (for each hole we remove all the vertices above that hole):



These Schubert varieties correspond to the permutations: (1237456) and (2341567) . Let $X(w')$ be a Schubert subvariety in $X(w)$ whose generic point is not in the Gorenstein locus. Then $X(w')$ has to be contained in $X(1237456)$.

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1 Minuscule Schubert varieties

Let us fix some notations and recall the definitions of minuscule homogeneous spaces and minuscule Schubert varieties. A basic reference is [LMS79].

In this paper G will be a semi-simple algebraic group, we fix B a Borel subgroup and T a maximal torus in B . We denote by R the set of roots, by R^+ and R^- the set of positive and negative roots. We denote by S the set of simple roots. We will denote by W the Weyl group of G .

We also fix P a parabolic subgroup containing B . We denote by W_P the Weyl group of P and by W^P the set of minimal length representatives in W of the coset W/W_P . Recall that the Schubert varieties in G/P (that is to say the B -orbit closures in G/P) are parametrised by W^P .

DEFINITION 1.1. — *A fundamental weight ϖ is said to be minuscule if, for all positive roots $\alpha \in R^+$, we have $\langle \alpha^\vee, \varpi \rangle \leq 1$.*

With the notation of N. Bourbaki [Bo68], the minuscule weights are:

Type	minuscule
A_n	$\varpi_1 \cdots \varpi_n$
B_n	ϖ_n
C_n	ϖ_1
D_n	ϖ_1, ϖ_{n-1} and ϖ_n
E_6	ϖ_1 and ϖ_6
E_7	ϖ_7
E_8	none
F_4	none
G_2	none

DEFINITION 1.2. — *Let ϖ be a minuscule weight and let P_ϖ be the associated parabolic subgroup. The homogeneous space G/P_ϖ is then said to be minuscule. The Schubert varieties of a minuscule homogeneous space are called minuscule Schubert varieties.*

REMARK 1.3. — *It is a classical fact that to study minuscule homogeneous spaces and their Schubert varieties, it is sufficient to restrict ourselves to simply-laced groups.*

In the rest of the paper, the group G will be simply-laced, the subgroup P will be a maximal parabolic subgroup associated to a minuscule fundamental weight ϖ . The minuscule homogeneous space G/P will be denoted by X and the Schubert variety associated to $w \in W^P$ will be denoted by $X(w)$ with the convention that the dimension of $X(w)$ is the length of w .

2 Miniscule quivers

In [Pe07], we associated to any minuscule Schubert variety $X(w)$ a unique quiver Q_w . The definition a priori depends on the choice of a reduced expression but does not depend on the commuting relations. In the minuscule setting this implies that the following definitions do not depend on the chosen reduced expression. Fix a reduced expression $w = s_{\beta_1} \cdots s_{\beta_r}$ of w (recall that w is in W^P the set of minimal length representatives of W/W_P) where for all $i \in [1, r]$, we have $\beta_i \in S$.

DEFINITION 2.1. — (i) *The successor $s(i)$ and the predecessor $p(i)$ of an element $i \in [1, r]$ are the elements $s(i) = \min\{j \in [1, r] \mid j > i \text{ and } \beta_j = \beta_i\}$ and $p(i) = \max\{j \in [1, r] \mid j < i \text{ and } \beta_j = \beta_i\}$.*

(ii) *Denote by Q_w the quiver whose set of vertices is the set $[1, r]$ and whose arrows are given in the following way: there is an arrow from i to j if and only if $\langle \beta_j^\vee, \beta_i \rangle \neq 0$ and $i < j < s(i)$ (or only $i < j$ if $s(i)$ does not exist).*

REMARK 2.2. — (i) *This quiver comes with a coloration of its vertices by simple roots via the map $\beta : [1, r] \rightarrow S$ such that $\beta(i) = \beta_i$.*

(ii) *There is a natural order on the quiver Q_w given by $i \preceq j$ if there is an oriented path from j to i . Caution that this order is the reversed order of the one defined in [Pe07].*

(iii) *Note that if we denote by Q_ϖ the quiver obtained from the longest element in W^P , then the quiver Q_w is a subquiver of Q_ϖ . The quivers of Schubert subvarieties are exactly the order ideals in the quiver Q_ϖ . We will call such a quiver reduced (meaning that it corresponds to a reduced expression of an element in W^P , see [Pe07] for more details on the shape of reduced quivers).*

Recall also that we defined in [Pe07] some combinatorial objects associated to the quiver Q_w .

DEFINITION 2.3. — (i) *We call peak any vertex of Q_w maximal for the partial order \preceq . We denote by $\text{Peaks}(Q_w)$ the set of peaks of Q_w .*

(ii) *We call hole of the quiver Q_w any vertex i of Q_ϖ satisfying one of the following properties*

- *the vertex i is in Q_w but $p(i) \notin Q_w$ and there are exactly two vertices $j_1 \succ i$ and $j_2 \succ i$ in Q_w with $\langle \beta_i^\vee, \beta_{j_k} \rangle \neq 0$ for $k = 1, 2$.*
- *the vertex i is not in Q_w , $s(i)$ does not exist in Q_ϖ and there exist $j \in Q_w$ with $\langle \beta_i^\vee, \beta_j \rangle \neq 0$.*

Because the vertex of the second type of holes is not a vertex in Q_w we call such a hole a virtual hole of Q_w . We denote by $\text{Holes}(Q_w)$ the set of holes of Q_w .

(iii) The height $h(i)$ of a vertex i is the largest positive integer n such that there exists a sequence $(i_k)_{k \in [1, n]}$ of vertices with $i_1 = 1$, $i_n = r$ and such that there is an arrow from i_k to i_{k+1} for all $k \in [1, n - 1]$.

Many geometric properties of the Schubert variety $X(w)$ can be read on its quiver. In particular we proved in [Pe07, Corollary 4.12]:

PROPOSITION 2.4. — *A Schubert subvariety $X(w')$ in $X(w)$ is stable under $\text{Stab}(X(w))$ if and only if $\beta(\text{Holes}(Q_{w'})) \subset \beta(\text{Holes}(Q_w))$.*

An easy consequence of this fact and the result by M. Brion and P. Polo that the smooth locus of $X(w)$ is the dense $\text{Stab}(X(w))$ -orbit is the following:

PROPOSITION 2.5. — *A Schubert variety $X(w)$ is smooth if and only if all the holes of its quiver Q_w are virtual.*

We will be more precise in Theorem 3.2 and we will describe the irreducible components of the singular locus and the generic singularity of this component in terms of the quiver. The Gorensteiness of the variety is also easy to detect on the quiver as we proved in [Pe07, Corollary 4.19]:

PROPOSITION 2.6. — *A Schubert variety $X(w)$ is Gorenstein if and only if all the peaks of its quiver Q_w have the same height.*

3 Generic singularities of minuscule Schubert varieties

In this section, we go one step further in the direction of reading on the quiver Q_w the geometric properties of $X(w)$. We will translate the results of M. Brion and P. Polo [BP99] on the irreducible components of the singular locus of $X(w)$ and the singularity at a generic point of such a component in terms of the quiver Q_w . We will need the following notations:

DEFINITION 3.1. — (i) Let i be a vertex of Q_w , we define the subquiver Q_w^i of Q_w as the full subquiver containing the following set of vertices $\{j \in Q_w / j \succ i\}$. We denote by $Q_{w,i}$ the full subquiver of Q_w containing the vertices of $Q_w \setminus Q_w^i$. We denote by w^i (resp. w_i) the elements in W^P associated to the quivers Q_w^i (resp. $Q_{w,i}$).

(ii) A hole i of the quiver Q_w is said to be essential if it is not virtual and if there is no hole in the subquiver Q_w^i .

(iii) Following M. Brion and P. Polo, denote by J the set $\beta(\text{Holes}(Q_w))^c$.

We then prove the following:

THEOREM 3.2. — (i) The set of irreducible components of the singular locus of $X(w)$ is in one to one correspondence with the set of essential holes of the quiver Q_w . In particular, if i is an essential hole of Q_w , the corresponding irreducible component is the Schubert subvariety $X(w_i)$ of $X(w)$ whose quiver is $Q_{w,i}$.

(iii) Furthermore, the singularity of $X(w)$ at a generic point of $X(w_i)$ is the same singularity as the one of the B -fixed point in the Schubert variety $X(w^i)$ whose quiver is Q_w^i .

REMARK 3.3. — The singularity of the B -fixed point in $X(w^i)$ is described in [BP99].

PROOF — This result is a reformulation of the main results of M. Brion and P. Polo [BP99]. Proposition 2.4 shows that the essential holes are in one to one correspondence with maximal Schubert subvarieties in $X(w)$ stable under $\text{Stab}(X(w))$ and that if i is an essential hole, then the corresponding Schubert subvariety $X(w_i)$ is associated to the quiver $Q_{w,i}$. According to [BP99], these are the irreducible components of the singular locus.

To describe the singularity of $X(w_i)$, M. Brion and P. Polo define two subsets I and I' of the set of simple roots as follows:

- the set I is the union of the connected components of $J \cap w_i(R_P)$ adjacent to $\beta(i)$
- the set I' is the union $I \cup \{\beta(i)\}$.

We describe these sets thanks to the quiver.

PROPOSITION 3.4. — The set I' is $\beta(Q_w^i)$.

PROOF — The elements in $J \cap w_i(R_P)$ are the simple roots $\gamma \in J$ such that $w_i^{-1}(\gamma) \in R_P$. Thanks to Lemma 3.5, these elements are the simple roots in J neither in $\beta(\text{Holes}(Q_{w,i}))$ nor in $\beta(\text{Peaks}(Q_{w,i}))$.

An easy (but fastidious for types E_6 and E_7) look on the quivers shows that $I' = \beta(Q_w^i)$. A uniform proof of this statement is possible but needs an involved case analysis on the quivers. \square

LEMMA 3.5. — Let β be a simple root, then we have

1. $w^{-1}(\beta) \in R^- \setminus R_P^-$ if $\beta \in \beta(\text{Peaks}(Q_w))$,
2. $w^{-1}(\beta) \in R^+ \setminus R_P^+$ if $\beta \in \beta(\text{Holes}(Q_w)) = J^c$ or
3. $w^{-1}(\beta) \in R_P^+$ otherwise.

PROOF — Let $w = s_{\beta_1} \cdot s_{\beta_r}$ be a reduced expression for w , we want to compute $w^{-1}(\beta) = s_{\beta_r} \cdots s_{\beta_1}(\beta)$. We proceed by induction and deal with the three cases at the same time.

1. Take first $\beta \in \beta(\text{Peaks}(Q_w))$, we may assume that $\beta_1 = \beta$ and $w^{-1}(\beta) = s_{\beta_r} \cdots s_{\beta_2}(-\beta)$. Let $i \in \text{Peaks}(Q_w)$ such that $\beta(i) = \beta$, the quiver obtained by removing i has $s(i)$ for hole (possibly virtual). We may apply induction and the result in case 2.

2.a. Let $\beta \in J^c$. Assume first that there is no $k \in Q_w$ with $\beta(k) = \beta$. Then there exist an $i \in Q_w$ such that $\langle \beta^\vee, \beta_i \rangle \neq 0$. Let us prove that such a vertex i is unique. Indeed, the support of w is contained in a subdiagram D of the Dynkin diagram not containing β . The diagram D contains the simple root α corresponding to P (except if $X(w)$ is a point in which case $w = \text{Id}$ and the lemma is easy). The quiver Q_w is in particular contained in the quiver of the minuscule

homogeneous variety associated to $\alpha \in D$. It is easy to check on these quivers (see in [Pe07] for the shape of these quivers) that there is a unique such vertex i .

Now consider the quivers Q_w^i and $Q_{w,i}$. Recall that we denote by w^i and w_i the associated elements in W . We have $w = w^i w_i$. We compute $w^{i-1}(\beta)$ and because all simple roots $\beta(x)$ for $x \in Q_w^i$ with $x \neq i$ are orthogonal to β we have $w^{i-1}(\beta) = s_{\beta_i}(\beta) = \beta + \beta_i$. We then have $w^{-1}(\beta) = w_i^{-1}(\beta + \beta_i)$. Because i was the only vertex such that $\langle \beta^\vee, \beta_i \rangle \neq 0$, we have $w_i^{-1}(\beta) = \beta \in R_P^+$ and by induction (note that i is now a hole of $Q_{w,i}$) we have $w_i^{-1}(\beta_i) \in R^+ \setminus R_P^+$ and we have the result.

2.b. Now assume that there exist $k \in \text{Holes}(Q_w)$ with $\beta(k) = \beta$ and let i a vertex maximal for the property $\langle \beta^\vee, \beta_i \rangle \neq 0$. Remark that we have $k < i$. Consider one more time the quivers Q_w^i and $Q_{w,i}$ and the elements w^i and w_i . We have $w^{-1}(\beta) = w_i^{-1}(\beta_i + \beta)$. But as before we have by induction $w_i^{-1}(\beta_i) \in R^+ \setminus R_P^+$ so that we can conclude by induction as soon as k is not a peak of $Q_{w,i}$. But because k is an hole, there exist a vertex $j \in Q_w$ with $j \neq i$ and such that there is an arrow $j \rightarrow k$ in Q_w . Because i was taken maximal j is a vertex of $Q_{w,i}$ and k is not a peak of this quiver.

3. If β is not in the support of w but is not in $\beta(\text{Holes})$ then $w^{-1}(\beta) = \beta \in R_P^+$.

Let β in $\beta(Q_w)$ but not in $\beta(\text{Holes}(Q_w))$ or $\beta(\text{Peaks}(Q_w))$ and let k the highest vertex such that $\beta(k) = \beta$. There exists a unique vertex $i \in Q_w$ such that $i \succ k$ and $\langle \beta^\vee, \beta_i \rangle \neq 0$. We have $w^{-1}(\beta) = w_i^{-1}(\beta_i + \beta)$ and the vertex k is a peak of $Q_{w,i}$ so that $w_i = s_{\beta(k)} w_k = s_\beta w_k$ and $w^{-1}(\beta) = w_k^{-1}(\beta_i)$. Now it is easy to see that either $s(i)$ does not exists and in this case it is not a virtual hole or it exists but is neither a peak nor a hole of $Q_{w,k}$. We conclude by induction on the third case. \square

The Theorem is now a corollary of the description of the singularities thanks to I and I' done by M. Brion and P. Polo. \square

REMARK 3.6. — *In their article M. Brion and P. Polo also deal with the cominucule Schubert varieties. We believe that, in that case, Theorem 0.3 should hold true as well as Corollary 0.4.*

It is now easy to decide which generic singularity is Gorenstein:

COROLLARY 3.7. — *Let i be an essential hole of the quiver Q_w . The generic point of the irreducible component $X(w_i)$ of the singular locus is Gorenstein if and only if all the peaks of Q_w^i are of the same height.*

We describe the Schubert subvarieties $X(w')$ in $X(w)$ that are expected to be Gorenstein at their generic point by the conjecture of A. Woo and A. Yong. Let us give the following

DEFINITION 3.8. — (i) *An essential hole is said to be Gorenstein if the generic point of the associated irreducible component of the singular locus is in the Gorenstein locus.*

(ii) *A Schubert subvariety $X(w')$ in $X(w)$ is said to have the property (WY) if the generic point of any irreducible component of the singular locus of $X(w)$ containing $X(w')$ is in the Gorenstein locus of $X(w)$.*

We have the following:

PROPOSITION 3.9. — *Let $X(w')$ be a Schubert subvariety of the Schubert variety $X(w)$. If the generic point of $X(w')$ is Gorenstein in $X(w)$, then $X(w')$ has the property (WY).*

PROOF — Let $X(v)$ be an irreducible component of the singular locus of $X(w)$ containing $X(w')$. Because the property of being non Gorenstein is stable under closure, this implies that the generic point of $X(v)$ is Gorenstein in $X(w)$. \square

Remark that, because all the irreducible components of the singular locus of $X(w)$ are stable under $\text{Stab}(X(w))$, the property (WY) need only to be checked on $\text{Stab}(X(w))$ -stable Schubert subvarieties.

PROPOSITION 3.10. — *(i) The Schubert subvarieties $X(w')$ in $X(w)$ stable under $\text{Stab}(X(w))$ are exactly those such that the associated quiver $Q_{w'}$ satisfies*

$$Q_{w'} = \bigcap_{i \in \text{Holes}(Q_w)} Q_{w, s^{k_i}(i)}$$

where the $(k_i)_{i \in \text{Holes}(Q_w)}$ are integers greater or equal to -1 (if $k_i = -1$, the quiver $Q_{w, s^{k_i}(i)}$ is Q_w by definition).

(ii) A $\text{Stab}(X(w))$ -stable Schubert subvariety $X(w')$ of $X(w)$ has the property (WY) if and only if the only essential holes in the difference $Q_w \setminus Q_{w'}$ are Gorenstein. Equivalently, writing

$$Q_{w'} = \bigcap_{i \in \text{Holes}(Q_w)} Q_{w, s^{k_i}(i)},$$

if and only if the only holes in of the quivers $(Q_w^{s^{k_i}(i)})_{i \in \text{Holes}(Q_w)}$ are Gorenstein holes. Another equivalent formulation is that $Q_{w'}$ contains all the non Gorenstein essential holes of Q_w .

PROOF — (i) Consider the subquiver $Q_{w'}$ in Q_w and for each hole i of Q_w define the integer $k_i = \min\{k \geq 0 / s^k(i) \in Q_{w'}\} - 1$. Because of the fact (see for example [LMS79]) that the strong and weak Bruhat orders coincide for minuscule Schubert varieties, the quiver $Q_{w'}$ has to be contained in the intersection

$$Q' = \bigcap_{i \in \text{Holes}(Q_w)} Q_{w, s^{k_i}(i)}.$$

We therefore need to remove some vertices to Q' to get $Q_{w'}$. But removing a vertex j of the quiver Q' (it has to be a peak of Q') creates a hole in $s(j)$ (or a virtual hole in j if $s(j)$ does not exist). Because $X(w')$ is $\text{Stab}(X(w))$ -stable, the last removed vertex j is such that $\beta(j) \in \beta(\text{Holes}(Q_w))$. This implies that no more vertex can be removed from Q' to get $Q_{w'}$ and in particular $Q_{w'} = Q'$.

(ii) The Schubert subvariety has the property (WY) if and only if all the irreducible components $X(w_i)$ of the singular locus of $X(w)$ containing $X(w')$ are such that i is a Gorenstein hole. But $X(w')$ is contained in $X(w_i)$ if and only if $Q_{w'}$ is contained in $Q_{w,i}$. This is equivalent to the fact that Q_w^i is contained in $Q_w \setminus Q_{w'}$ and the proof follows. \square

4 Relative canonical model and Gorenstein locus

In this section, we recall the explicit construction given in [Pe07] of the relative canonical model of $X(w)$. Recall that we described in [Pe07] the Bott-Samelson resolution $\pi : \tilde{X}(w) \rightarrow X(w)$ as a configuration variety à la Magyar [Ma98]:

$$\tilde{X}(w) \subset \prod_{i \in Q_w} G/P_{\beta_i}$$

where P_{β_i} is the maximal parabolic associated to the simple root β_i . The map $\pi : \tilde{X}(w) \rightarrow X(w)$ is given by the projection $\prod_{i \in Q_w} G/P_{\beta_i} \rightarrow G/P_{\beta_{m(w)}}$ where $m(w)$ is the smallest element in Q_w .

We define a partition on the peaks of the quiver Q_w and a partition of the quiver itself:

DEFINITION 4.1. — (i) Define a partition $(A_i)_{i \in [1, n]}$ of $\text{Peaks}(Q_w)$ by induction: A_1 is the set of peaks with minimal height and A_{i+1} is the set of peaks in $\text{Peaks}(Q_w) \setminus \bigcup_{k=1}^i A_k$ with minimal height (the integer n is the number of different values the height function takes on the set $\text{Peaks}(Q_w)$).

(ii) Define a partition $(Q_w(i))_{i \in [1, n]}$ of Q_w by induction:

$$Q_w(i) = \{x \in Q_w \mid \exists j \in A_i : x \preccurlyeq j \text{ and } x \not\preccurlyeq k \ \forall k \in \bigcup_{j > i} A_j\}.$$

We proved in [Pe07] that these quivers $Q_w(i)$ are quivers of minuscule Schubert varieties and in particular have a minimal element $m_w(i)$. We defined the variety $\hat{X}(w)$ as the image of the Bott-Samelson resolution $\tilde{X}(w)$ (seen as a configuration variety) in the product $\prod_{i=1}^n G/P_{\beta_{m_w(i)}}$.

Because $m_w(n) = m(w)$ we have a map $\hat{\pi} : \hat{X}(w) \rightarrow X(w)$ and a factorisation

$$\begin{array}{ccc} \tilde{X}(w) & \xrightarrow{\tilde{\pi}} & \hat{X}(w) \\ & \searrow \pi & \downarrow \hat{\pi} \\ & & X(w). \end{array}$$

We proved the following result in [Pe07]:

THEOREM 4.2. — (i) The variety $\hat{X}(w)$ together with the map $\hat{\pi}$ realise $\hat{X}(w)$ as the relative canonical model of $X(w)$.

(ii) The variety $\hat{X}(w)$ is a tower of locally trivial fibrations with fibers the Schubert varieties associated to the quivers $Q_w(i)$. In particular $\hat{X}(w)$ is Gorenstein.

We will use this resolution to prove our main result. Indeed, we will prove that the generic fibre of the map $\hat{\pi} : \hat{X}(w) \rightarrow X(w)$ above a (WY) Schubert subvariety $X(w')$ is a point. In other words, the map $\hat{\pi}$ is an isomorphism on an open subset of $X(w')$. As a consequence, the generic point of $X(w')$ will be in the Gorenstein locus.

Let us recall some facts on $\tilde{X}(w)$ and $\hat{X}(w)$ (see [Pe07]):

FACT 4.3. — (i) To each vertex i of Q_w one can associate a divisor D_i on $\tilde{X}(w)$ and all these divisors intersect transversally.

(*n*) For K a subset of the vertices of Q_w , we denote by Z_K the transverse intersection of the D_i for $i \in K$.

(*m*) The image of the closed subset Z_K by the map π is the Schubert variety $X(w_K)$ whose quiver Q_{w_K} is the biggest reduced subquiver of Q_w not containing the vertices in K .

The quiver $Q_w(i)$ defines a element $w(i)$ in W and the fact that these quivers realise a partition of Q_w implies that we have an expression $w = w(1) \cdots w(n)$ with $l(w) = \sum l(w(i))$. We prove the following generalisation of this fact:

PROPOSITION 4.4. — *Let K be a subset of the vertices of Q_w . The image of the closed subset Z_K by the map $\tilde{\pi}$ is a tower of locally trivial fibrations with fibers the Schubert varieties $X(w_K(i))$ whose quiver $Q_{w_K(i)}$ is the biggest reduced subquiver of $Q_{w(i)}$ not containing the vertices of $K \cap Q_{w(i)}$.*

This variety is the image by $\tilde{\pi}$ of $Z_{\cup_{i=1}^n Q_K(i)}$.

PROOF — As we explained in [Pe07, Proposition 5.9], the Bott-Samelson resolution is the quotient of the product $\prod_{i \in Q_w} R_i$ where the R_i are certain minimal parabolic subgroups by a product of Borel subgroups $\prod_{i=1}^r B_i$. The variety $\hat{X}(w)$ is the quotient of a product $\prod_{i=1}^n N_i$ of parabolic subgroups such that the multiplication in G maps $\prod_{k \in Q_{w(i)}} R_k$ to N_i by a product $\prod_{i=1}^n M_i$ of parabolic subgroups. The map $\tilde{\pi}$ is induced by the product from $\prod_{i \in Q_w} R_i$ to $\prod_{i=1}^n N_i$. In particular, this means that for $i \in [1, n]$ fixed, the map $\prod_{k \in Q_{w(i)}} R_k \rightarrow N_i$ induces the map from the Bott-Samelson resolution $\tilde{X}(w(i))$ to $X(w(i))$. We may now apply part (*m*) of the preceding fact because the quiver $Q_w(i)$ is minuscule. \square

We now remark that the quivers $Q_{w'}$ associated to Schubert subvarieties $X(w')$ in the Schubert variety $X(w)$ having the property (WY) have a nice behaviour with respect to the partition $(Q_w(i))_{i \in [1, n]}$ of Q_w .

PROPOSITION 4.5. — *Let $X(w')$ be a $\text{Stab}(X(w))$ -stable Schubert subvariety of $X(w)$ having the property (WY). Let us denote by $(C_j)_{j \in [1, k]}$ the connected components of the subquiver $Q_w \setminus Q_{w'}$ of Q_w . Then for each j , there exist a unique $i_j \in [1, n]$ such that $C_j \subset Q_w(i_j)$.*

PROOF — Recall from Proposition 3.10 that, denoting by $\text{GorHol}(Q_w)$ the set of Gorenstein holes in Q_w , we may write

$$Q_w \setminus Q_{w'} = \bigcup_{i \in \text{GorHol}(Q_w)} Q_w^{s^{k_i}(i)}$$

with k_i an integer greater or equal to -1 and with the additional condition that $Q_w^{s^{k_i}(i)}$ contains only Gorenstein holes. Because the quivers $Q_w^{s^{k_i}(i)}$ are connected, any connected component of $Q_w \setminus Q_{w'}$ is an union of such quivers. But we have the following:

LEMMA 4.6. — *Let $i \in \text{Holes}(Q_w)$ and assume that $Q_w^{s^k(i)}$ meets at least two subquivers of the partition $(Q_w(i))_{i \in [1, n]}$, then $Q_w^{s^k(i)}$ contains a non Gorenstein hole.*

PROOF — The quiver $Q_w^{s^k(i)}$ meets two subquivers of the partition $(Q_w(i))_{i \in [1, n]}$, in particular it contains two peaks of Q_w of different heights. By connexity of $Q_w^{s^k(i)}$, we may assume that these

two peaks are adjacent. In particular there is a hole between these two peaks and this hole is not Gorenstein and is contained in $Q_w^{s^k(i)}$. \square

The proposition follows. \square

We describe the inverse image by $\hat{\pi}$ of a $\text{Stab}(X(w))$ -stable Schubert subvariety of $X(w)$ having the property (WY). To do this, first remark that the map π is B -equivariant and that the inverse image $\pi^{-1}(X(w'))$ has to be a union of closed subsets Z_K for some subsets K of Q_w . Let $Z_K \subset \pi^{-1}(X(w'))$ be such that $\pi : Z_K \rightarrow X(w')$ is dominant. We will denote by $Q_w^{w'}(i)$ the intersection $Q_{w'} \cap Q_w(i)$ and by $w'(i)$ the associated element in W .

PROPOSITION 4.7. — *The image of Z_K in $\hat{X}(w)$ by $\tilde{\pi}$ is the same as the image of $Z_{Q_w \setminus Q_{w'}}$.*

PROOF — Thanks to Proposition 4.4 we only need to compute the quivers $Q_{w_K(i)}$. Consider the decomposition into connected components $Q_w \setminus Q_{w'} = \cup_{j=1}^k C_j$. We may decompose K accordingly as $K = \cup_{j=1}^k K_j$ where $K_j = K \cap C_j$. But because each connected component of $Q_w \setminus Q_{w'}$ is contained in one of the quivers $(Q_w(i))_{i \in [1, n]}$ this implies that $Q_{w_K(i)}$ is exactly $Q_{w_K} \cap Q_w(i)$ where Q_{w_K} is the biggest reduced quiver in $Q_w \setminus Q_{w'}$ not containing the vertices in K (see Fact 4.3). We get $Q_{w_K} = Q_{w'}$ (because Z_K is sent onto $X(w')$) and the result follows. \square

THEOREM 4.8. — *Let $X(w')$ be a Schubert subvariety in $X(w)$. Then $X(w')$ has the property (WY) if and only if its generic point is in the Gorenstein locus of $X(w)$.*

PROOF — We have already seen in Proposition 3.9 that if the generic point of $X(w')$ is in the Gorenstein locus of $X(w)$ then $X(w')$ has the property (WY).

Conversely let $X(w')$ be a Schubert subvariety having the property (WY). The previous proposition implies that its inverse image $\hat{\pi}^{-1}(X(w'))$ is the variety $\tilde{\pi}(Z_{Q_w \setminus Q_{w'}})$. But this implies that the map $\hat{\pi} : \tilde{\pi}(Z_{Q_w \setminus Q_{w'}}) = \hat{\pi}^{-1}(X(w')) \rightarrow X(w')$ is birational (because the varieties have the same dimension given by the number of vertices in the quiver). In particular, the map $\hat{\pi}$ is an isomorphism on an open subset of $X(w)$ meeting $X(w')$ non trivially. Therefore, because $\hat{X}(w)$ is Gorenstein, it is the case of the generic point in $X(w')$ as a point in $X(w)$. \square

References

- [BL00] *Sara Billey and Venkatramani Lakshmibai*: Singular loci of Schubert varieties. Progress in Mathematics, 182. Birkhäuser Boston, Inc., Boston, MA, 2000.
- [BW03] *Sara Billey and Gregory Warrington*: Maximal singular loci of Schubert varieties in $\text{SL}(n)/B$. Trans. Amer. Math. Soc. 355 (2003), no. 10, 3915–3945.
- [Bo68] *Nicolas Bourbaki*: Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV : Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines. Actualités Scientifiques et Industrielles, No. 1337 Hermann, Paris 1968.
- [BP99] *Michel Brion and Patrick Polo*: Generic singularities of certain Schubert varieties. Math. Z. 231 (1999), no. 2, 301–324.

- [Co03] *Aurélié Cortez*: Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire. Adv. Math. 178 (2003), no. 2, 396–445.
- [KLR03] *Christian Kassel, Alain Lascoux and Chritophe Reutenauer*: The singular locus of a Schubert variety. J. Algebra 269 (2003), no. 1, 74–108.
- [LMS79] *Venkatramani Lakshmibai, Chitikila Musili and Conjeerveram S. Seshadri*: Geometry of G/P . III. Standard monomial theory for a quasi-minuscule P . Proc. Indian Acad. Sci. Sect. A Math. Sci. 88 (1979), no. 3, 93–177.
- [LS90] *Venkatramani Lakshmibai and B. Sandhya*: Criterion for smoothness of Schubert varieties in $Sl(n)/B$. Proc. Indian Acad. Sci. Math. Sci. 100 (1990), no. 1, 45–52.
- [Ma98] *Peter Magyar*: Borel-Weil theorem for configuration varieties and Schur modules. Adv. Math. 134 (1998), no. 2, 328–366.
- [Ma01a] *Laurent Manivel*: Le lieu singulier des variétés de Schubert. Internat. Math. Res. Notices 2001, no. 16, 849–871.
- [Ma01b] *Laurent Manivel*: Generic singularities of Schubert Varieties. math.AG/0105239 (2001).
- [Pe07] *Nicolas Perrin*: Small-resolutions of minuscule Schubert varieties. Preprint math.AG/0601117 to appear in Compositio Mathematica (2007).
- [WY06a] *Alexander Woo and Alexander Yong*: When is a Schubert variety Gorenstein? Adv. Math. 207 (2006), no. 1, 205–220.
- [WY06b] *Alexander Woo and Alexander Yong*: Governing singularities of Schubert varieties. Preprint math.AG/0603273 (2006).

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